COMPLEX NUMBERS

If 'a', 'b' are two real numbers, then a number of the form a + ib is called a complex number

Set of complex Numbers : The set of all complex numbers is denoted by C.

i.e. $C = \{a + ib \mid a, b \in R\}$

Equality of Complex Numbers : Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal if $a_1 = a_2$ and $b_1 = b_2$ i.e. Re (z_1) = Re(z_2) and Im (z_1) = Im (z_2)

FUNDAMENTAL OPERATIONS ON COMPLEX NUMBERS

ADDITION : Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then their sum $z_1 + z_2$ is defined as the complex number $(a_1 + a_2) + i(b_1 + b_2)$

Properties of addition of complex numbers

(i) Addition is commutative : For any two complex numbers z_1 and z_2, we have

$$z_1 + z_2 = z_2 + z_1$$

(ii) Addition is associative : For any three complex numbers z_1, z_2, z_3 we have

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$$

(iii) Existence of additive identity : The complex number 0 = 0 + i0 is the identity element for addition i.e.

$$z + 0 = z = 0 + z \text{ for all } z \in C$$

(iv) Existence of additive inverse : For every complex number z there exists –z such that

$$z + (-z) = 0 = (-z) + z$$

The complex number –z is called the additive inverse of z.

Substraction : Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then the subtraction of z_2 from z_1 is denoted by $z_1 - z_2$ and is defined as the addition of z_1 and $-z_2$.

Thus, $z_1 - z_2$

$= (a_1 - a_2) + i(b_1 - b_2)$

Multiplication : Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ be two complex numbers. Then, the multiplication of z_1 with z_2 is denoted by z_1z_2 and is defined as the complex number.

$$(a_1a_2 - b_1 b_2) + i(a_1b_2 + a_2b_1)$$

Properties of Multiplication :

(i) Multiplication is commutative. For any two complex numbers z_1 and z_2, we have

$$z_1 z_2 = z_2 z_1$$

(ii) Multiplication is associative : For any three complex numbers z_1, z_2, z_3 we have

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

(iii) Existence of identity element for multiplication. The complex number 1 = 1 + i0 is the identity element for multiplication i.e. for every complex number z, we have

$$z . 1 = z$$

(iv) Exitence of multiplicative inverse : Corresponding to every non-zero complex number z = a + ib there exists a complex number $z_1 = x + iy$ such that

$$z . z_1 = 1 \Rightarrow z_1 = \frac{1}{z}$$

The complex number z_1 is called the multiplicative inverse or reciprocal of z and is given by

$$z_1 = \frac{a}{a^2+b^2} + \frac{i(-b)}{a^2+b^2}$$

(v) Multiplication of complex numbers is distributive over addition of complex numbers : For any three complex numbers z_1, z_2, z_3 we have

(i) $z_1(z_2+z_3) = z_1z_2 + z_1z_3$ (Left distributivity)

(ii) $(z_2+z_3)z_1 = z_2z_1 + z_3z_1$ (Right distributivity)

Division : The division of a complex number z_1 by a non-zero complex number z_2 is defined as the multiplication of z_1 by the multiplicative inverse of z_2 and is denoted by $\frac{z_1}{z_2}$.

Thus, $\frac{z_1}{z_2} = z_1.z_2^{-1} = z_1.\left(\frac{1}{z_2}\right)$

Conjugate : Let z = a + ib be a complex number. Then the conjugate of z is denoted by $\bar{z}$ and is equal to a – ib.

Thus, $z = a + ib \Rightarrow \bar{z} = a - ib$

Properties of Conjugate :

If z, z_1, z_2 are complex numbers, then

(i) $z + \bar{z} = 2\,\mathrm{Re}(z)$

(ii) $z - \bar{z} = 2\,\mathrm{Im}(z)$

(iii) $z = \bar{z} \Leftrightarrow z$ is purely real

(iv) $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary.

(v) $z\bar{z} = \{\mathrm{Re}(z)\}^2 + \{(\mathrm{Im}(z)\}^2$

(vi) $\overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$

(vii) $\overline{z_1-z_2} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{z_1\ z_2} = \bar{z}_1\ \bar{z}_2$

(x) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}, z_2 \neq 0$

(xi) $\overline{(\bar{z})} = z$

MODULUS OF A COMPLEX NUMBER

Definition : the modulus of a complex number z = a + i b is denoted by |z| and is defined as

$$|z| = \sqrt{a^2+b^2} = \sqrt{\{\mathrm{Re}(z)\}^2 + \{\mathrm{Im}(z)\}^2}$$

- The multiplicative inverse of a non-zero complex number z is same as its reciprocal and is given by

$$\frac{\mathrm{Re}(z)}{|z|^2} + i\frac{(-\mathrm{Im}(z))}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

If b is positive

then $\therefore \sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\left\{\sqrt{a^2+b^2}+a\right\}} + i\sqrt{\frac{1}{2}\{\sqrt{a^2+b^2}-a\}}\right]$

If b is negative then

$$\sqrt{a+ib} = \pm\left[\sqrt{\frac{1}{2}\{|z|+\text{Re}(z)\}} - i\sqrt{\frac{1}{2}\{|z|-\text{Re}(z)\}}\right]$$

Argument or (amplitude) of a Complex Number

(i) If x and y both are positive, then the argument of z = x + iy is the acute angle given by $\tan^{-1}\left|\frac{y}{x}\right|$

(ii) x < 0 and y > 0, then the argument of z = x + iy is $\pi - \alpha$, where α is the acute angle given by $\tan^{-1}|y/x|$.

(iii) If x < 0 and y < 0 then the argment of z = x + iy is $\alpha - \pi$ where α is the acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$.

(iv If x > 0 and y <0, then the argument of z = x + iy is $-\alpha$ where α s the acute angle given by $\tan\alpha = \left|\frac{y}{x}\right|$

Polar or Trigonometrical Form of a Complex Number

Let z = x + iy be a complex number represented by a point P (x, y) in the Argand plane. Then, by the geometrical representation of z = x + i y, we have

$$\Rightarrow z = r(\cos\theta + i\sin\theta), \text{where } r = |z| \text{ and } \theta = \arg(z)$$

This form of z is called a polar form of z.

EULERIAN FORM OF A COMPLEX NUMBER

$e^{i\theta} = \cos\theta + i\sin\theta$ and $e^{-i\theta} = \cos\theta - i\sin\theta$

Properties of Argument of z

(i) $\arg(\bar{z}) = -\arg(z)$

(ii) $\arg(z_1z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg(z_1\bar{z}_2) = \arg(z_1) - \arg(z_2)$

(iv) $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$

(v) $\arg(z^n) = n \arg z$.

(vi) $|z_1+z_2| = |z_1-z_2| \Rightarrow \arg z_1 - \arg z_2 = \pi/2$

(vii) $|z_1+z_2| = |z_1|+|z_2| \Rightarrow \arg z_1 = \arg z_2$

(viii) If arg z = 0, then z is purely real

(ix) If $\arg z = \pm\pi/2$, then z is purely imaginary

Properties of Modulus of z

(i) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1-\theta_2)$.

or

$|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(ii) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

or

$|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,\mathrm{Re}(z_1\bar{z}_2)$

(iii) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2)$

(iv) $|z_1 + z_2| = |z_1 - z_2| \Rightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(v) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2)$

(vi) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(vii) $|z_1 + z_2| \le |z_1| + |z_2|$

(viii) $|z_1 - z_2| \le |z_1| + |z_2|$

(ix) $|z_1 + z_2| \ge |z_1| - |z_2|$

(x) $|z_1 - z_2| \ge \big||z_1| - |z_2|\big|$

(xi) $|z_1 z_2| = |z_1||z_2|$

(xii) $|z^n| = |z|^n$

(xiii) $|z|^2 = z\bar{z}$

(xiv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

<u>Distance Between Two Points :</u> If z_1 and z_2 are the affixes of points P and Q respectively in the argand plane, then

$$PQ = |z_2 - z_1|$$

<u>Section Formula :</u> Let z_1 and z_2 be the affixes of two points P and Q respectively in the argand plane. Then, the affix of a point R dividing PQ internally in the ratio m : n is

$$\frac{mz_2 + nz_1}{m+n}$$ but if R is external point, then affix of R is $$\frac{mz_2 - nz_1}{m-n}$$

<u>Mid Point Formula :</u>

If R be the mid-point then affix of R is $\frac{z_1 + z_2}{2}$

- If z_1, z_2, z_3 are affixes of the vertices of a triangle, then the affix of its centroid is

$$\frac{z_1 + z_2 + z_3}{3}$$

- The equation of the perpendicular bisector of the line segment joining points having affixes z_1 and z_2 is

$$z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$$

- The equation of a circle whose centre is at point having affix z_0 and radius R is

$$|z - z_0| = R$$

<u>Note :</u>
- If the centre of the circle is at the origin and radius R, then its equation is $|z| = R$.
- General Equation of circle is

$z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ where $b \in R$ and a is complex number

represents a circle having centre at '–a'

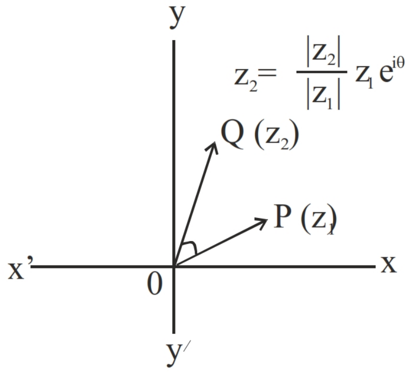
and radius $= \sqrt{|a|^2 - b} = \sqrt{a\bar{a} - b}$

COMPLEX NUMBER AS A ROTATING ARROW IN THE ARGAND PLANE

To obtain the point representing $ze^{i\alpha}$ we rotate. $\overrightarrow{OP}$ through angle α in anticlock wise sense. Thus, multiplication by $e^{i\alpha}$ to z rotates the vector $\overrightarrow{OP}$ in anticlockwise sense through an angle α.

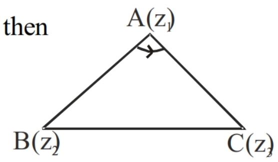
Let z_1 and z_2 be two complex numbers represented by points P and Q in the argand plane such that $\angle POQ = \theta$.

Then, $z_1e^{i\theta}$ is a vector of magnitude $|z_1|$ = OP along $\overrightarrow{OQ}$ and $\frac{z_1e^{i\theta}}{|z_1|}$ is a unit vector along $\overrightarrow{OQ}$.



SOME IMPORTANT RESULTS

I. If z_1, z_2, z_3 are the affixes of the points A, B and C in the Argand plane, then



(i) $\angle BAC = \arg\left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

(ii) $\angle BAC = \arg\frac{z_3 - z_1}{z_2 - z_1} = \frac{|z_3 - z_1|}{|z_2 - z_1|}(\cos\alpha + i\sin\alpha)$, where $\alpha = \angle BAC$.

If z_1, z_2, z_3 and z_4 are the affixes of the points A, B, C and D respectively in the Argand plane. Then AB is inclined to CD at the angle.

$$\arg\left(\frac{z_2 - z_1}{z_4 - z_3}\right)$$

(iii) The equation of the circle having z_1 and z_2 as the end points of a diameter is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (\bar{z} - \bar{z}_1)(z - z_2) = 0$$

DE-MOIVERE'S THEOREM

STATEMENT :

(i) If $n \in z$ (the set of integers), then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

(ii) If $n \in Q$ (the set of rational numbers), then $\cos n\theta + i\sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$.

(iii) $\frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$

(iv) $(\cos\theta_1 + i\sin\theta_2)(\cos\theta_2 + i\sin\theta_2) = \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

n^{th} ROOTS OF UNITY

nth roots of unity are : $\alpha^0 = 1, \alpha, \alpha^2, \alpha^3, ...\alpha^{n-1}$ where $\alpha = e^{i2\pi/n} = \cos 2\pi/n + i \sin 2\pi/n$

PROPERTIES OF n^{th} ROOTS OF UNITY

Property 1 : nth roots of unity form a G.P. with common ratio $e^{i2\pi/n}$
Property 2 : Sum of the nth rots of unity is always zero.
Property 3 : Sum of pth powers of nth roots of unity is zero, if p is not a multiple of n.
Property 4 : Sum of pth powers of nth roots of unity is n, if p is a multiple of n.
Property 5 : Product of nth roots of unity is $(-1)^{n-1}$
Property 6 : nth roots of unity lie on the unit circle |z| = 1 and divide its circumference into n equal parts.

* PROPERTIES OF CUBE ROOTS OF UNITY AND SOME USEFUL RESULTS RELATED TO THEM

(i) Cube roots of unity are $1, \omega, \omega^2$ where

$$\omega = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

(ii) $\arg(\omega) = 2\pi/3$ and

(iii) Cube roots of –1 are –1, $-\omega, -\omega^2$

(iv) $1 + \omega + \omega^2 = 0$

(v) $\omega^3 = 1$

* Four fourth roots of unity are –1, 1, –i, i

* $\log(\alpha + i\beta) = \frac{1}{2}\log(\alpha^2 + \beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$

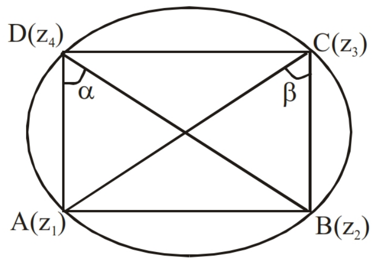
* Condition for points $A(z_1)$, $B(z_2)$, $C(z_3)$, $D(z_4)$ to be concyclic :

$$\alpha = \beta$$

$$\Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4}\right) = \arg\left(\frac{z_2 - z_3}{z_1 - z_3}\right)$$

$$\Rightarrow \arg\left(\frac{z_2 - z_4}{z_1 - z_4} \times \frac{z_1 - z_3}{z_2 - z_3}\right) = 0$$

$$\Rightarrow \left(\frac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)}\right) \text{ is purely real.}$$

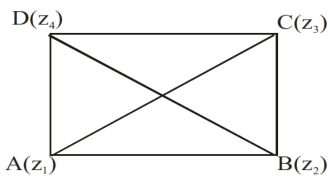


Condition (s) for four points $A(z_1), B(z_2), C(z_3)$ and $D(z_4)$ to represent vertices of a

(I) Parallelogram :

(i) The diagonards AC and BD must bisect each other

$$\Leftrightarrow \frac{1}{2}(z_1 + z_3) = \frac{1}{2}(z_2 + z_4)$$



$\Leftrightarrow z_1 + z_3 = z_2 + z_4$

(ii) Rhombus :

(a) The diagonads AC and BD bisect each other.

$\Leftrightarrow z_1 + z_3 = z_2 + z_4$

and (b) a pair of two adjacent sides are equal i.e. AD = AB.

$\Leftrightarrow |z_4 - z_1| = |z_2 - z_1|$

(iii) Square :

(a) The diagonals AC and BD bisects each other

$\Leftrightarrow z_1 + z_3 = z_2 + z_4$

(b) a pair of adjacent sides are equal

AD = AB

$\Leftrightarrow |z_4 - z_1| = |z_2 - z_1|$

(c) The two diagonards are equal

AC = BD $\quad \Leftrightarrow |z_3 - z_1| = |z_4 - z_2|$

(iv) Rectangle :

(a) The diagonals AC and BD bisect each other

$z_1 + z_3 = z_2 - z_4$

(b) The diagonalds AC and BD are equal

$\Leftrightarrow |z_3 + z_1| = |z_4 - z_2|$

(v) Incentre : I (z) of the ΔABC is given by

$$z = \frac{az_1 + bz_2 + cz_3}{a + b + c}$$

(vi) Circumcentre (z) of the ΔABC is given by

$$z = \frac{z_1(\sin 2A) + z_2(\sin 2B) + z_3(\sin 2C)}{\sin 2A + \sin 2B + \sin 2C}$$

(v) Orthocentre (z) of the ΔABC is given by

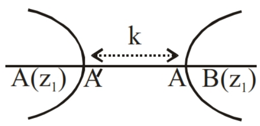
$$z = \frac{(a \sec A)z_1 (b \sec B)z_2 + (c\ \sec C)z_3}{a \sec A + b \sec B + c \sec C}$$

(vi) Area of triangle ABC with vertices $A(z_1)$, $B(z_2)$, $C(z_3)$ is given by

$$\Delta = \frac{1}{4} \text{ modulus of } \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

(vi) Equation of line passing through $A(z_1)$ and $B(z_2)$ is

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$



(vii) General equation of a line is

$\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

(viii) Complex slope of a line joining points A(z_1) and B(z_2) is

given by $w = \dfrac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$

(ix) Two lines with complex slopes w_1 and w_2 are parallel if $w_1 = w_2$ and perpendicular if $w_1 + w_2 = w$

Length of perpendicular from a point to a line

Length of perpendicular of a point A(ω) from the line $\bar{a}z + a\bar{z} + b = 0$

$$p = \frac{|a\omega + a\bar{\omega} + b|}{2|a|}$$

Recongnizing some loci by Inspection :

(i) If z_1 and z_2 are two fixed points, then

$|z - z_1| = |z - z_2|$ represent perpendicular bisector of the line segment joining A(z_1) andB(z_2).

(ii) If z_1 and z_2 are two fixed points and k >0, $k \neq 1$ is a real number then $\dfrac{|z - z_1|}{|z - z_2|} = k$ represents a circle. For $k = 1$ it represents perpendicular bisector of the segment joining A(z_1) and B(z_2)

(iii) Let z_1 and z_2 be two fixed points and k be a positive real number.

(a) If $k > |z_1 - z_2|$ then $|z - z_1| + |z - z_2| = k$ represents an ellipse with foci at A(z_1) and B(z_2) and length of major axis = k = CD.

(b) If $k = |z_1 - z_2|$ represents the line segment joining z_1 and z_2.

(c) If $k < |z_1 - z_2|$ then $|z - z_1| + |z - z_2| = k$ does not represent any curve in the argand plane.

(iv) Let z_1 and z_2 be two fixed points, k be a positive real number.

(a) If $k < |z_1 - z_2|$, then $|z - z_1| - |z - z_2| = k$ represents a hyperbola with foci at A(z_1) and B(z_2).

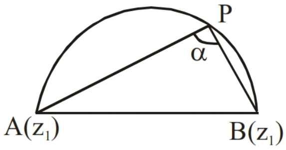
(b) If $k = (z_1 - z_2)$, then

$(z - z_1) - (z - z_2) = k$

represents the straight line joining A(z_1) and B(z_2) excluding the segment AB.

(v) If z_1 and z_2 are two fixed points, then $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$ represents a circle with z_1 and z_2 as extremities of a diameter.

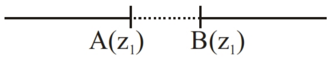
(vi) Let z_1 and z_2 be two fixed points and α be a real number such that $0 \leq \alpha \leq \pi$ then (a) If $0 < \alpha < \pi$ and $\alpha \neq \dfrac{\pi}{2}$ then $\arg\left(\dfrac{z - z_1}{z - z_2}\right) = \alpha$ represents a segment of the circle passing through A(z_1) and B(z_2)



(b) If $\alpha_2 = \pi/2$, then $\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha = \frac{\pi}{2}$ represents a circle with diameter as the segment joining

$A(z_1)$ and $B(z_2)$.

(c) if $\alpha = \pi$ then $\arg\left(\frac{z-z_1}{z-z_2}\right) = \alpha$ represents the straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB.



(d) If $\alpha = 0$, then $\arg\left(\frac{z-z_1}{z-z_2}\right) - \alpha(=0)$

represents the segment joining $A(z_1)$ and $B(z_2)$